DEMONSTRATIO MATHEMATICA
Vol. XXXVIII No 4 2005

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FINITE SEQUENCES OF SUBSHIFTS OF FINITE TYPE

Abstract. The goal of this paper is to present some dynamical properties of finite sequences of finite type subshifts. Moreover, some conditions for algorithmization of these properties are formulated.

1. Introduction

A subshift is a space consisting of a collection of infinite symbol sequences that are usually taken from a finite set. Finite type subshifts (vertex subshifts) generated by square matrices with nonnegative entries are a special sort of discrete dynamical systems, often called symbolic dynamical systems [4, 9, 11, 15]. By using the symbolic dynamics we can often analyse the dynamics of nonlinear and chaotic systems. The dynamics of hyperbolic systems can be described by subshifts of finite type. An essential example of the applications of subshifts is the Smale's horseshoe, which is topologically conjugate to the full shift on two symbols. To analyse properties of the dynamical system which is topologically conjugate to a subshift of finite type it is enough to analyse dynamical properties of the subshift. Mañé proved a theorem (for finite type subshifts generated by $0,1$-square matrices), stating that topological transitivity and topological mixing can be reduced to algebraic properties of associated matrices [10]. In papers [6], [7] there were presented necessary and sufficient conditions for classical subshift of finite type to be a nonempty set. By applying the concept of topological transitivity—which is equivalent to the existence of a dense orbit—one of the chaos definitions has been formulated and in this sense of this definition the subshifts of finite type generated by irreducible matrices are chaotic [4]. In paper [8] conditions for algorithmization of topological transitivity and topological mixing of the subshift of finite type were provided.

1991 *Mathematics Subject Classification*: Primary 65P20, Secondary 37M99.
Key words and phrases: subshift of finite type, topological transitivity, topological mixing, finite sequence subshifts of finite type.

The aim of this paper is to provide the reader with algorithmizable conditions equivalent to topological transitivity and topological mixing of finite sequences of subshifts of finite type. The paper is composed of three sections. In the first section we give preliminaries concerning the basic facts of dynamics of finite type subshifts. In section two is given the notion of finite ascending sequences of finite type subshifts. Moreover, in this section conditions for algorithmization of nonemptiness problem of finite sequences of subshifts are formulated. Analogously, to dynamical properties of subshifts of finite type we can analyse dynamical properties of the finite ascending sequences of finite type subshifts.

2. Preliminaries

Let us denote by $\mathcal{M}_n(0,1)$ the set of all $(0,1)$-square matrices of size n.

DEFINITION 2.1. The oriented graph $G(A)$ associated with a matrix $A \in \mathcal{M}_n(0,1)$ consists of n vertices. There is an edge in $G(A)$ from the i-th to the j-th vertex iff $a_{ij} = 1$, where $A = [a_{ij}]_{i,j=1,\dots,n}$.

DEFINITION 2.2. A graph is said to be an *irreducible* (*strongly connected*) iff for every two vertices v_1, v_2 there is a path between vertices v_1, v_2.

DEFINITION 2.3. The space

$$\Sigma_n := \{x = \{x_i\}_{i\in\mathbb{Z}} : x \in \{1,\dots,n\}^{\mathbb{Z}}\}.$$

with the shift mapping $\sigma : \Sigma_n \to \Sigma_n$ defined by

$$\sigma(x)_i = x_{i+1}.$$

is called *the full shift on n symbols.*

DEFINITION 2.4. Having given a matrix $A \in \mathcal{M}_n(0,1)$ let us define

$$\Sigma_A := \{x \in \Sigma_n : \ \forall_{i\in\mathbb{Z}}\, a_{x_i x_{i+1}} = 1\}.$$

OBSERVATION 2.5. The space Σ_A with the Tichonov topology is compact and invariant under σ.

The pair $(\Sigma_A, \sigma|_{\Sigma_A})$ is called a *subshift of finite type.*

Let X be a compact metric space.

DEFINITION 2.6. Let X be a compact metric space. A point $x \in X$ is said to be a *nonwandering point* for a homeomorphism $f : X \to X$ iff for every open neighbourhood U of x there is a positive integer m such that $f^{-m}(U) \cap U \neq \emptyset$.

The set of all nonwandering points of f will be denoted by $\Omega(f)$.

DEFINITION 2.7. We call f *topologically transitive* iff for all open nonempty sets $U, V \subset X$ there is a positive integer m such that $f^{-m}(U) \cap V \neq \emptyset$.

DEFINITION 2.8. We call f *topologically mixing* iff for all open nonempty sets $U, V \subset X$ there is a positive integer m_0 such that $f^{-m}(U) \cap V \neq \emptyset$ for every integer $m > m_0$.

DEFINITION 2.9. A closed invariant set $U \subset X$ is said to be a *basic set* iff the homeomorphism $f|_U$ is topologically transitive.

DEFINITION 2.10. A matrix $A \in \mathcal{M}_n(0,1)$ is said to be a *transition matrix* iff there is a nonzero entry in every column and every row of A.

From now on, having given a matrix A, by a_{ij}^m we shall denote the term of the matrix A^m on the i,j-th position.

DEFINITION 2.11. A is *irreducible* iff for every $i,j \in \{1,\dots,n\}$ there is a positive integer m such that $a_{ij}^m > 0$.

DEFINITION 2.12. A is *mixing* iff for every $i,j \in \{1,\dots,n\}$ there is a positive integer m_0 such that $a_{ij}^m > 0$ for every integer $m > m_0$.

THEOREM 2.13 (see [6]). *Let $A = [a_{ij}]_{i,j=1,\dots,n} \in \mathcal{M}_n(0,1)$. Then the following conditions are equivalent:*

(i) $\Sigma_A \neq \emptyset$;

(ii) *there exist $k \in \{1,\dots,n\}$ and pairwise distinct integers $b_1,\dots,b_k \in \{1,\dots,n\}$ such that $D = [a_{b_i b_j}]_{i,j=1,\dots,k} \in \mathcal{E}_k$;*

(iii) $\operatorname{tr}(A + A^2 + \dots + A^n) > 0$.

OBSERVATION 2.14. $\operatorname{tr}(A + A^2 + \dots + A^n) > 0 \Leftrightarrow$ *in the graph $G(A)$ there is a cycle.*

Define $B = A + A^2 + \dots + A^n$. Then:

OBSERVATION 2.15. $\Sigma_A \neq \emptyset \iff \operatorname{tr}(B) > 0$.

THEOREM 2.16 (see [8]). *For a given transition matrix A the following are true:*

(1) $\sigma|_{\Sigma_A}$ *is topologically transitive* $\iff b_{ij} > 0 \ \forall_{i,j \in \{1,\dots,n\}}$;

(2) $\sigma|_{\Sigma_A}$ *is topologically mixing* $\iff A$ *is irreducible and*

$$1 \in \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{Z} \cdot j \cdot \operatorname{sgn}(a_{ii}^j),$$

where $\mathbb{Z}$ is the set of integers. The equivalent statement of the above condition is

$$\exists \, \{c_{ij}\}_{i,j \in \{1,\dots,n\}} \subset \mathbb{Z} \ : \ 1 = \sum_{i,j=1}^{n} c_{ij} \cdot j \cdot \operatorname{sgn}(a_{ii}^j).$$

OBSERVATION 2.17. $\forall_{i,j \in \{1,\dots,n\}} \exists_{m>0} : a_{ij}^m > 0 \iff \forall_{i,j \in \{1,\dots,n\}} b_{ij} > 0$.

Let $\mathrm{GCD}(m_1,\dots,m_k)$ be the *greatest common divisor* of positive integers $m_1,\dots,m_k$. Since there exist integers $t_1,\dots,t_k$ such that

$$\mathrm{GCD}(m_1,\dots,m_k) = t_1m_1 + t_2m_2 + \cdots + t_km_k,$$

we can give the following corollary.

COROLLARY 2.18. *Let A be a transition matrix. Then: $\sigma|_{\Sigma_A}$ is topologically mixing $\Longleftrightarrow$ A is irreducible and there exist cycles $\pi_1,\dots,\pi_k$ of lengths $m_1,\dots,m_k \le n$ in the graph $G(A)$ such that*

$$\mathrm{GCD}(m_1,\dots,m_k) = 1.$$

Therefore, if there exists a positive integer k such that

$$\prod_{l=1}^{k} a_{i_l i_l}^{m_l} > 0, \;\; \mathrm{GCD}(m_1,\dots,m_k) = 1, \;\; i_l, m_l \in \{0,\dots n\},$$

then the map $\sigma|_{\Sigma_A}$ is topologically mixing.

In paper [7] there are finite type subshifts generated by a finite sum of square matrices defined and analysed. Let $A_1,\dots,A_k \in \mathcal{M}_n(0,1)$ generate subshifts $\Sigma_{A_1},\dots,\Sigma_{A_k}$. Let us consider the subshift Σ_C, where

$$C = [c_{ij}], \;\; c_{ij} = \mathrm{sgn}\,((a_1)_{ij} + \cdots + (a_k)_{ij}).$$

COROLLARY 2.19. $\Sigma_{A_1} \cup \cdots \cup \Sigma_{A_k} \subset \Sigma_C$.

THEOREM 2.20 (see for instance [7]). *Under the above assumptions, if one of the finite type subshifts $\Sigma_{A_1},\dots,\Sigma_{A_k}$ is nonempty, then $\Sigma_C \ne \emptyset$.*

COROLLARY 2.21. *If*

$$\mathrm{tr}\,(A_1+A_1^2+\cdots+A_1^n)+\mathrm{tr}\,(A_2+A_2^2+\cdots+A_2^n)+\cdots+\mathrm{tr}\,(A_k+A_k^2+\cdots+A_k^n) > 0,$$

then $\Sigma_C \ne \emptyset$.

COROLLARY 2.22. *Let $A_1,\dots,A_k$ be the transition matrices defined above.*

1. *If one of the maps $\sigma|_{\Sigma_{A_1}},\dots,\sigma|_{\Sigma_{A_k}}$ is topologically transitive then the map $\sigma|_{\Sigma_C}$ is topologically transitive.*
2. *If one of the maps $\sigma|_{\Sigma_{A_1}},\dots,\sigma|_{\Sigma_{A_k}}$ is topologically mixing then the map $\sigma|_{\Sigma_C}$ is topologically mixing.*

COROLLARY 2.23. *If*

$$(A_1+A_1^2+\cdots+A_1^n)+(A_2+A_2^2+\cdots+A_2^n)+\cdots+(A_k+A_k^2+\cdots+A_k^n) > 0,$$

then the map $\sigma|_{\Sigma_C}$ is topologically transitive.

COROLLARY 2.24. *If one of the transition matrices $A_1,\dots,A_k$ is irreducible and the condition $\mathrm{tr}\,(A_1 + A_2 + \cdots + A_k) > 0$ is satisfied then the map $\sigma|_{\Sigma_C}$ is topologically mixing.*

EXAMPLE 2.25. Let

$$A_1 = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0&0&0&1 \\ 1&1&1&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0&0&1&0 \\ 1&0&0&0 \\ 0&1&1&1 \\ 0&1&0&0 \end{bmatrix},$$

then

$$C = \operatorname{sgn}(A_1 + A_2 + A_3) = \begin{bmatrix} 0&1&1&1 \\ 1&1&1&1 \\ 0&1&1&1 \\ 0&1&0&1 \end{bmatrix}.$$

The condition $\operatorname{tr}(A_1) \neq 0$ holds and the matrix A_3 is the sum of the irreducible matrix A_4 and the matrix A_5, where

$$A_4 = \begin{bmatrix} 0&0&1&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&1&0 \\ 0&0&0&0 \end{bmatrix}.$$

Basing on the above theorems and corollaries $\Sigma_C \neq \emptyset$ and the map $\sigma|_{\Sigma_C}$ is topologically transitive and topologically mixing.

3. Finite ascending sequences of finite type subshifts

Let

$$k_1 < k_2 < \cdots < k_s, \quad s \in \mathbb{N},$$

be nonnegative integers and let

$$A_1 \in \mathcal{M}_{k_1}(0,1),\ A_2 \in \mathcal{M}_{k_2}(0,1)\ \ldots, A_s \in \mathcal{M}_{k_s}(0,1),$$

be square matrices where

$$A_c = [(a_c)]_{i,j=1,\ldots,k_c}, \quad c = 1, \ldots, s.$$

Moreover, let the conditions

$$(a_c)_{ij} = (a_{c-1})_{ij}, \ \text{ for } i,j \in \{1, \ldots, k_{c-1}\}$$

be satisfied. In this paper we consider *a finite ascending sequence of finite type subshifts*

$$\Sigma_{A_1}, \ \Sigma_{A_2}, \ \ldots, \ \Sigma_{A_s}.$$

Since matrices A_1, A_2, $\ldots$, A_s satisfy the above conditions, sequence

$$\Sigma_{A_1}, \ \Sigma_{A_2}, \ \ldots, \ \Sigma_{A_s}$$

is joined, what means

$$\Sigma_{A_1} \subset \Sigma_{A_2} \subset \cdots \subset \Sigma_{A_s} \subset \Sigma_{k_s}.$$

In some causes it is possible that

$$\Sigma_{A_1} = \Sigma_{A_2} = \cdots = \Sigma_{A_s}.$$

EXAMPLE 3.1. Let

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\Sigma_{A_1} = \Sigma_{A_2} = \Sigma_{A_3} = \Sigma_1$.

EXAMPLE 3.2. Let

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then $\Sigma_{A_1} = \Sigma_1, \ \Sigma_{A_2} = \Sigma_4, \ \Sigma_{A_3} = \Sigma_6$.

THEOREM 3.3. *Let $\Sigma_{A_1}, \Sigma_{A_2}, \ldots, \Sigma_{A_s}$ be a finite ascending sequence of finite type subshifts. If the condition*

$$\operatorname{tr}(A_{\bar{c}} + A_{\bar{c}}^2 + \cdots + A_{\bar{c}}^{k_{\bar{c}}}) > 0$$

is satisfied for some $\bar{c} \in \{1, \ldots, s\}$, then $\Sigma_{A_c} \neq \emptyset$ for $c \in \{\bar{c}, \ldots, s\}$.

Proof. If the condition

$$\operatorname{tr}(A_{\bar{c}} + A_{\bar{c}}^2 + \cdots + A_{\bar{c}}^{k_{\bar{c}}}) > 0$$

is true for some $\bar{c} \in \{1, \ldots, s\}$ then by Observation 2.14 there is a cycle π in the graph $G(A)$. As the sequence of subshifts is ascending then

$$(a_c)_{ij} = (a_{c-1})_{ij}, \quad \text{for } i, j \in \{1, \ldots, k_{c-1}\}.$$

As a consequence, the cycle π is the cycle in graphs associated with matrices A_c for $c \in \{\bar{c}, \ldots, s\}$. The condition

$$\operatorname{tr}(A_c + A_c^2 + \cdots + A_c^{k_c}) > 0$$

is true for $c \in \{\bar{c}, \ldots, s\}$ and $\Sigma_{A_c} \neq \emptyset$ for $c \in \{\bar{c}, \ldots, s\}$.

COROLLARY 3.4. *Let $\Sigma_{A_1}, \Sigma_{A_2}, \ldots, \Sigma_{A_s}$ be the finite ascending sequence of finite type subshifts. If the condition $\Sigma_{A_{\bar{c}}} \neq \emptyset$ is true for some $\bar{c} \in \{1, \ldots, s\}$, then $\Sigma_{A_c} \neq \emptyset$ for $c \in \{\bar{c}, \ldots, s\}$.*

Let $G_1, G_2, \dots, G_s$ be graphs associated with matrices $A_1, A_2, \dots, A_s$. Moreover, let $V_1, V_2, \dots, V_s$ and $E_1, E_2, \dots, E_s$ be the sets of vertices and edges of above graphs, respectively. Then graph G_i is a subgraph of G_j ($G_i \subset G_j$) for $i, j \in A_s$, $i < j$, if the inclusions $V_i \subset V_j$, $E_i \subset E_j$ are true.

COROLLARY 3.5. *If matrices $A_1, A_2, \dots, A_s$ are generated by a finite ascending sequence of finite type subshifts, then $G_1 \subset \dots \subset G_s$.*

Let Σ_{A_1}, Σ_{A_1}, $\dots$, Σ_{A_s} be a finite ascending sequence of finite type subshifts. We create auxiliary matrices

$$\bar{A}_1 \in \mathcal{M}_{k_1}(0,1), \;\; \bar{A}_2 \in \mathcal{M}_{k_2-k_1}(0,1), \;\; \dots, \; \bar{A}_s \in \mathcal{M}_{k_s-k_{s-1}}(0,1)$$

defined in the following way

$$(\bar{a}_c)_{(i-k_{c-1})(j-k_{c-1})} = (a_c)_{ij}$$
$$\text{for } \; i,j \in \{k_{c-1}+1, \dots, k_c\}, \; c \in \{1, \dots, s\}, \; k_0 = 0.$$

LEMMA 3.6. *Let Σ_{A_1}, Σ_{A_2} be a finite ascending sequence of finite type subshifts. If the matrices $\bar{A}_1, \bar{A}_2$ are irreducible and the condition*

$$(a_2)_{i^1 j^1} \cdot (a_2)_{i^2 j^2} > 0, \;\; i^1, \, j^2 \in \{k_1+1, \dots, k_2\}, \;\; i^2, \, j^1 \in \{1, \dots, k_1\}$$

is satisfied then the matrices A_1, A_2 are irreducible.

Proof. Since $A_1 = \bar{A}_1$ it is enough to prove that A_2 is irreducible. We show that the graph $G(A_2)$ is irreducible. If $i,j \in \{1, \dots, k_1\}$ or $i,j \in \{k_1+1, \dots, k_2\}$, then there is a path between vertices i and j in the graph $G(A_2)$. It is easy to notice that there is a path between vertices i and j in the graph $G(A_2)$, where $i \in \{1, \dots, k_1\}$, $j \in \{k_1+1, \dots, k_2\}$. Because matrices $\bar{A}_1, \bar{A}_2$ are irreducible, it is easy to see, that there exists a path π_1, π_2 between vertices i, i^2 and j^2, j. The condition $(a_2)_{i^2 j^2} > 0$ implies that there is a path π_3 between vertices i_2, j_2. The path $\pi = \pi_1 \pi_3 \pi_2$ joins vertices i, j. Analogously, it can be proved that there is a path between vertices j and i. ∎

THEOREM 3.7. *Let Σ_{A_1}, Σ_{A_2}, $\dots$, Σ_{A_s} be a finite ascending sequence of finite type subshifts. If the matrices*

$$\bar{A}_1 \in \mathcal{M}_{k_1}(0,1), \; \bar{A}_2 \in \mathcal{M}_{k_2-k_1}(0,1), \; \dots, \; \bar{A}_s \in \mathcal{M}_{k_s-k_{s-1}}(0,1),$$

are irreducible and the condition

$$(a_s)_{i_c^1 j_c^1} \cdot (a_s)_{i_c^2 j_c^2} > 0 \;\; \textit{for some}$$
$$i_c^1, j_c^2 \in \{k_c+1, \dots, k_{c+1}\}, \; i_c^2, j_c^1 \in \{k_{c-1}+1, \dots, k_c\}$$

for all $c \in \{1, \dots, s\}$ where $k_0 = 0$ is satisfied then the map $\sigma|_{\Sigma_{A_c}}$ is topologically transitive.

Proof. If the matrices $\bar{A}_1, \ldots, \bar{A}_s$ are irreducible and the condition

$$(a_s)_{i_c^1 j_c^1} \cdot (a_s)_{i_c^2 j_c^2} > 0 \text{ for some } i_c^1, j_c^2 \in \{k_c + 1, \ldots, k_{c+1}\},\ i_c^2, j_c^1 \in \{k_{c-1} + 1, \ldots, k_c\}$$

for all $c \in \{1, \ldots, s\}$ is satisfied, then by the above lemma and by induction, the irreducibility of matrices $A_1, \ldots, A_s$ is proved and the maps $\sigma|_{\Sigma_{A_1}}, \ldots \sigma|_{\Sigma_{A_s}}$ are topologically transitive. ■

THEOREM 3.8. *Let $\Sigma_{A_1}, \Sigma_{A_2}, \ldots, \Sigma_{A_s}$ be a finite ascending sequence of finite type subshifts. Let the matrices*

$$\bar{A}_1 \in \mathcal{M}_{k_1}(0,1), \bar{A}_2 \in \mathcal{M}_{k_2 - k_1}(0,1), \ \ldots, \ \bar{A}_s \in \mathcal{M}_{k_s - k_{s-1}}(0,1),$$

be irreducible and there exist cycles $\pi_1, \ldots, \pi_p$ in graphs

$$G(\bar{A}_c),\ c \in \{\bar{s}_1, \ldots, \bar{s}_2\},\ 0 \le \bar{s}_1 \le \bar{s}_2 \le s$$

of lengths ξ_l, $l \in \{1, \ldots, p\}$, such that

$$\mathrm{GCD}\,(\xi_1, \ldots, \xi_p) = 1.$$

Then, if the condition

$$(a_s)_{i_c^1 j_c^1} \cdot (a_s)_{i_c^2 j_c^2} > 0 \ \text{for}\ i_c^1, j_c^2 \in \{k_c + 1, \ldots, k_{c+1}\},\ i_c^2, j_c^1 \in \{k_{c-1} + 1, \ldots, k_c\}$$

for all $c \in \{1, \ldots, s\}$, where $k_0 = 0$ is satisfied then the map $\sigma|_{\Sigma_{A_c}}$ is topologically mixing for $c \in \{\bar{s}_2, \ldots, s\}$.

Proof. If the matrices $\bar{A}_1, \ldots, \bar{A}_s$ are irreducible, then the matrices $A_1, \ldots, A_s$ are irreducible, too (the above lemma). Since there exist cycles $\pi_1, \ldots, \pi_p$ of lengths η_l, $l \in \{1, \ldots, p\}$ in graphs $G(\bar{A}_c)$, where $c \in \{\bar{s}_1, \ldots, \bar{s}_2\}$, $0 \le \bar{s}_1 \le \bar{s}_2 \le s$, such that

$$\mathrm{GCD}\,(\eta_1, \ldots, \eta_p) = 1,$$

then the matrix A_{s_1} is mixing and as a consequence matrices $A_{s_1}, \ldots, A_s$ are mixing, too (Theorem 2.16). The theorem is proved. ■

COROLLARY 3.9. *Let $\Sigma_{A_1}, \Sigma_{A_2}, \ldots, \Sigma_{A_s}$ be a finite ascending sequence of finite type subshifts. If the maps*

$$\sigma|_{\bar{A}_1},\ \sigma|_{\bar{A}_2},\ \ldots, \sigma|_{\bar{A}_s}$$

are topologically mixing and the condition

$$(a_s)_{i_c^1 j_c^1} \cdot (a_s)_{i_c^2 j_c^2} > 0 \quad \text{for}\ i_c^1, j_c^2 \in \{k_c + 1, \ldots, k_{c+1}\},\ i_c^2, j_c^1 \in \{k_{c-1} + 1, \ldots, k_c\}$$

for all $c \in \{1, \ldots, s\}$, where $k_0 = 0$ is satisfied, then the map $\sigma|_{\Sigma_{A_c}}$ is topologically mixing for $c \in \{1, \ldots, s\}$.

Proof. If the maps

$$\sigma|_{\bar{A}_1},\ \sigma|_{\bar{A}_2},\ \ldots,\sigma|_{\bar{A}_s}$$

are topologically mixing then matrices $\bar{A}_1,\ldots,\bar{A}_s$ are mixing. Using of the above theorem ends the proof. ∎

COROLLARY 3.10. *Let $\Sigma_{A_1},\ \Sigma_{A_2},\ \ldots,\ \Sigma_{A_s}$ be a finite ascending sequence of finite type subshifts. If the matrices*

$$\bar{A}_1 \in \mathcal{M}_{k_1}(0,1),\ \ \bar{A}_2 \in \mathcal{M}_{k_2-k_1}(0,1),\ \ \ldots,\ \bar{A}_s \in \mathcal{M}_{k_s-k_{s-1}}(0,1),$$

are irreducible and there exists $\bar{s} \in \{1,\ldots,s\}$, such that

$$\operatorname{tr}(\bar{A}_{\bar{s}}) > 0,$$

then, if the condition

$(a_s)_{i_c^1 j_c^1}\cdot(a_s)_{i_c^2 j_c^2} > 0$ for $i_c^1, j_c^2 \in \{k_c+1,\ldots,k_{c+1}\}$, $i_c^2, j_c^1 \in \{k_{c-1}+1,\ldots,k_c\}$ *for all $c \in \{1,\ldots,s\}$, where $k_0 = 0$ is satisfied then the map $\sigma|_{\Sigma_{A_c}}$ is topologically mixing for $c \in \{\bar{s},\ldots,s\}$.*

EXAMPLE 3.11. Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix},\quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

be matrices which induce a finite sequence $\Sigma_{A_1}, \Sigma_{A_2}, \Sigma_{A_3}$. Since the matrices

$$\bar{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad \bar{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\quad \bar{A}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

are irreducible

$$(a_3)_{14}\cdot(a_3)_{41} > 0,\quad (a_3)_{75}\cdot(a_3)_{56} > 0,$$

and there exist cycles of lengths 2 and 3 in graphs $G(A_1)$, $G(A_2)$, the maps $\sigma|_{\Sigma_{A_2}}, \sigma|_{\Sigma_{A_3}}$ are topologically mixing.

THEOREM 3.12. *Let $\Sigma_{A_1},\ \Sigma_{A_2},\ \ldots,\ \Sigma_{A_s}$ be a finite ascending sequence of finite type subshifts such that*

$$A_i = \operatorname{Diag}(\bar{A}_1,\ldots,\bar{A}_i)\ \text{ for }\ i \in \{1,\ldots,s\}.$$

If the conditions

$$\bar{A}_1 + \bar{A}_1^2 + \cdots + \bar{A}_1^{k_1} > 0, \quad \bar{A}_2 + \bar{A}_2^2 + \cdots + \bar{A}_2^{k_2-k_1} > 0, \quad \ldots, \quad \bar{A}_s + \bar{A}_s^2 + \cdots + \bar{A}_s^{k_s-k_{s-1}} > 0$$

are satisfied, then

$$\Omega_i = \Sigma_{\bar{A}_1} \cup \cdots \cup \Sigma_{\bar{A}_i} \text{ for } i \in \{1, \ldots, s\},$$

where Ω_i is nonwandering set of the map $\sigma|_{\Sigma_i}$. Moreover, $\Sigma_{\bar{A}_1}, \ldots, \Sigma_{\bar{A}_i}$ are basic sets.

Proof. If the inequalities

$$\bar{A}_1 + \bar{A}_1^2 + \cdots + \bar{A}_1^{k_1} > 0, \quad \bar{A}_2 + \bar{A}_2^2 + \cdots + \bar{A}_2^{k_2-k_1} > 0, \quad \ldots, \quad \bar{A}_s + \bar{A}_s^2 + \cdots + \bar{A}_s^{k_s-k_{s-1}} > 0$$

hold, then matrices $\bar{A}_1, \ldots, \bar{A}_s$ are irreducible (Observation 2.17). So, the maps $\sigma|_{\bar{A}_1}, \ldots, \sigma|_{\bar{A}_s}$ are topologically transitive and we have exactly s disjoint strongly connected components. ∎

Let

$$\Sigma_{A_1}^1, \ldots, \Sigma_{A_s}^1, \ldots \Sigma_{A_1}^d, \ldots \Sigma_{A_s}^d,$$

be d-ascending sequences of finite type subshifts, where $d \in \mathbb{N}$. Let us consider a finite sequence $\Sigma_{C_1}, \ldots, \Sigma_{C_s}$, where

$$C_i = \operatorname{sgn}\left(A_i^1 + \cdots + A_i^d\right)$$

and $i \in \{1, \ldots, s\}$. In the analogous way as Corollary 2.23 we can formulate

COROLLARY 3.13. *Let $\Sigma_{A_1}^1, \ldots, \Sigma_{A_s}^1, \ldots \Sigma_{A_1}^d, \ldots \Sigma_{A_s}^d$, be d-ascending sequences of finite type subshifts. Let $\bar{s} = \min_{i=1,\ldots,s}\{i : B_i > 0\}$, where*

$$B_i = \operatorname{tr}(A_i^1 + \cdots + (A_i^1)^n) + \cdots + \operatorname{tr}(A_i^d + \cdots + (A_i^d)^n),$$

then $\Sigma_{C_i} \neq \emptyset$ for $i \in \{\bar{s}, \ldots, s\}$.

The proof is simple and we omit it.

REMARK. There exist another definitions of finite type subshifts, but the given above. The subject of this work are vertex subshifts. Received results can be applied to edge subshifts. In some papers the subshift of finite type is defined by a finite set of words having finite length (the set of forbidden words) that can not occur in sequences belonging to a given subshift [11]. By block maps (that are homeomorphisms) vertex subshift, which is topologically conjugate to initial subshift associated with a finite set of forbidden words, can be found. Thanks to this, received results from this paper can be applied to edge subshifts and subshifts generated by the set of forbidden words.

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Received May 4, 2004.